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SIMULTANEOUS, SUCCESSIVE AND ALTERNATING DIRECTION ITERATION SCHEMES

by

J. Heller

October 1, 1958

Institute of Mathematical Sciences

NEW YORK UNIVERSITY

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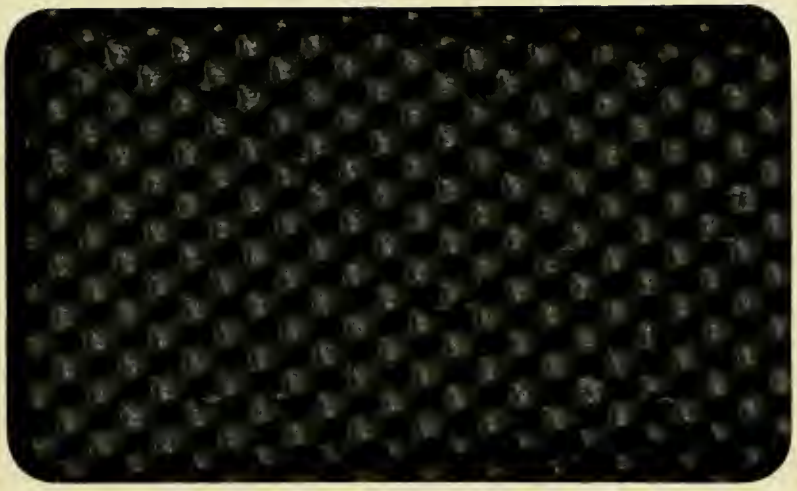

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INSTITUTE OF MATHEMATICAL SCIENCES
25 Waverly Place, New York 3, N. Y.

ABSTRACT

The linear systems approximating - in the lowest order of approximation - the 2pth order elliptic differential equation is derived. The connection between the simultaneous and successive iteration schemes is studied. In the case of difference equations in two dimensions which are separable the simultaneous, successive and some alternating direction iteration schemes are studied in detail. In particular the eigenvalues and eigenvectors of the error matrix are explicitly obtained.

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SIMULTANEOUS, SUCCESSIVE AND ALTERNATING DIRECTION ITERATION SCHEMES

1. Introduction.

The various iteration schemes for the solution of $2p^{th}$ order elliptic difference equations in two dimensions are studied from the point of view of minimizing a quadratic form and then from the point of view of separation of coordinates.

The general form of the matrix of the linear system is given and we note that there is always a possible partition of this matrix into triple diagonal blocks such that simultaneous and successive iteration schemes based on this partition will always converge. In all our examples we consider two dimensional meshes. The numbering of the mesh points differs somewhat from author to author so we will spell out an example in some detail. In a rectangle we number the horizontal mesh lines designated by $j = 0,1,2,\ldots,\nu$; while the vertical mesh lines are designated by $L = 0,1,2,\ldots,\mu$ (cf. fig. 1).

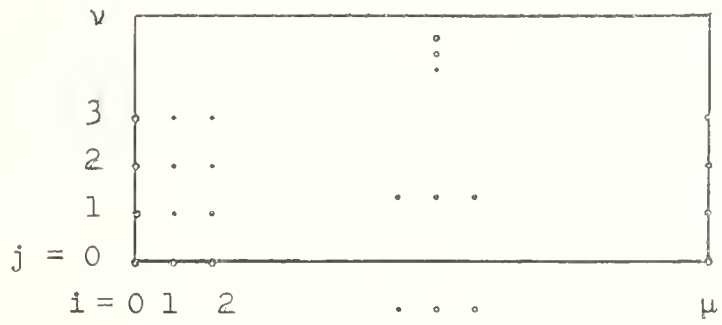


Figure 1

The unknowns of the difference equations u_{ij} will be components of a vector

$$U = \begin{pmatrix} u_{00} \\ u_{10} \\ \vdots \\ u_{\mu 0} \\ u_{01} \\ u_{11} \\ \vdots \\ u_{\mu 1} \\ \vdots \\ u_{0\nu} \\ u_{1\nu} \\ \vdots \\ u_{\mu\nu} \end{pmatrix}$$

This arrangement gives rise to a convenient matrix of the linear system. In the case of non-rectangular regions the mesh points are still arranged as above but each line will in general contain a different number of points.

Because very little can be said about the convergence rates of iteration schemes of general elliptic difference

equations of high order, we consider those difference equations in two dimensions in which the coordinates can be separated. From this point of view we study, for $2p^{th}$ order elliptic difference equations, the convergence rates of line simultaneous, line successive and alternating iteration schemes. In particular we derive the eigenvalues of the error matrix for each of these schemes for any order elliptic difference equations such that the coordinates are separable.[1]

[1] Some of these examples for second order elliptic difference equations were presented by Prof. B. Friedman in a seminar at N.Y.U.

2. Difference Equations Derived from Quadratic Form.

The derivation of elliptic difference equations from a quadratic form mimics the derivation of the elliptic differential equation from a variational principle. In deriving the difference equations that approximate an elliptic differential equation we approximate the variational integral by a quadratic plus linear difference form and minimize the difference form with respect to the unknowns, u_i. The variables u_i defined at mesh points approximate the solution of the elliptic differential equation at points p_i.

The quadratic form is

$$2Q = U^T AU - X^T B \tag{2.1}$$

where U is the vector of unknowns u_i, A is a positive definite symmetric matrix approximating, by differences, the quadratic portion of the variational integral and the vector B is the difference approximation to the linear form term of the variational integral.

For second order elliptic difference equations the matrix A is of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 & \cdots & & \\ A_{21} & A_{22} & A_{23} & 0 & \cdots & \\ 0 & A_{32} & A_{33} & A_{34} & 0 & \cdots \\ & & & \vdots & & \\ 0 & & \cdots & & 0 & A_{n-1\,n} \quad A_{nn} \end{pmatrix} \tag{2.2}$$

where n is the number of points in the one-dimensional case, the number of lines of points in a two-dimensional case, etc. In one dimension the elements A_{ij} are scalars, i.e. the weights approximating the second derivative. In two dimensions the elements A_{ij} are matrices. The diagonal elements A_{ii} represent the coupling along a line of mesh points and the off-diagonal elements represent the coupling with the line of points (see for example [4]). (In general for an n-dimensional region the diagonal elements A_{ii} are matrices coupling an (n-1)-dimensional manifold, and the off-diagonal elements represent the coupling with the remaining 1-dimensional manifolds of the n-dimensional space.)

For fourth order elliptic difference equations the matrix A is of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & \cdots & & \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & \cdots & & \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & 0 & \cdots & \\ 0 & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & 0 & \cdots \\ & & & & \ddots & \ddots & & \end{pmatrix} \tag{2.3}$$

In one dimension the elements A_{ij} are scalars and represent the five point approximation to the fourth derivative. In two dimensions the diagonal elements A_{ii} represent the coupling along a line of mesh points and the off

diagonal elements represent the coupling off the line of points; for the simplest approximation the diagonal elements A_{ii} are five diagonal, the super- and sub-diagonal elements $A_{i\ i-1}$ and $A_{i\ i+1}$ are three diagonal and the remaining elements A_{i-2i} and $A_{i\ i+2}$ are diagonal (see for example [3, 5, 6]).

In general for the $2p^{th}$ order elliptic difference equations the matrix A is of the form

(2.4) [See p. 10.]

If we partition the matrix A such that the diagonal matrices are $p \times p$ matrices with elements A_{ij}, except possibly the last block on the diagonal, we obtain a form for A which is triple diagonal:

$$(2.5) \qquad A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & & & \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & & \\ 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots & \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 & \cdots \\ \vdots & & & \ddots & & & \\ & & & & 0 & a_{m-1m-2} & a_{m-1m-1} & a_{m-1m} \\ 0 & & & & 0 & 0 & a_{m\ m-1} & a_{mm} \end{pmatrix}$$

We derive the difference equations by setting the variation of Q given by (2.1) equal to zero for arbitrary variations δu_i. The linear equations are

$$
(2.4)\qquad A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & \cdots & A_{1\,p+1} & 0 & \cdots & & & \\
A_{21} & A_{22} & A_{23} & \cdots & A_{2\,p+1} & A_{2\,p+2} & 0 & \cdots & & \\
A_{31} & A_{32} & A_{33} & \cdots & A_{3\,p+1} & A_{3\,p+2} & A_{3\,p+3} & 0 & \cdots & \\
\vdots & & & \ddots & & & & \ddots & & \\
A_{p+1\,1} & A_{p+1\,2} & \cdots & & A_{p+1\,p+1} & \cdots & & & A_{p+1\,2p+1} & 0 & \cdots \\
0 & A_{p+2\,2} & A_{p+2\,3} & \cdots & & A_{p+2\,p+2} & & & A_{p+2\,2p+1} & A_{p+2\,2p+2} & 0 & \cdots \\
\vdots & & & \ddots & & \ddots & & & & & \ddots & \\
0 & 0 & 0 & \cdots & & & A_{n\,n-p-1} & \cdots & & A_{n\,n-1} & A_{nn}
\end{pmatrix}
$$

(2.6) $$AU = B$$

A block successive iteration scheme based on the partitioned form of A given by (2.5)

(2.7)

$$\begin{pmatrix} a_{11} & 0 & \cdots & & \\ a_{21} & a_{22} & 0 & \cdots & \\ 0 & a_{32} & a_{33} & 0 & \cdots \\ & \ddots & & & \\ & & a_{m-2\,m-1} & a_{m-1\,m-1} & 0 \\ & & & a_{m\,m-1} & a_{mm} \end{pmatrix} {}^{n+1}U = B - \begin{pmatrix} 0 & a_{12} & 0 & \cdots & \\ 0 & 0 & a_{23} & 0 & \cdots \\ & & \ddots & & \\ 0 & \cdots & & 0 & a_{m-1\,m} \\ 0 & \cdots & & & 0 \end{pmatrix} {}^{n}U,$$

which is one of the many possible block successive iteration schemes, will converge [1], since A is symmetric positive definite. Defining the error as

(2.8) $$\varepsilon^{(n)} = U - U^{(n)} ,$$

we obtain from (2.6) and (2.7)

(2.9)

$$\begin{pmatrix} a_{11} & 0 & \cdots & \\ a_{21} & a_{22} & 0 & \cdots \\ 0 & a_{32} & a_{33} & 0 & \cdots \\ & \ddots & & & \end{pmatrix} \varepsilon^{(n+1)} = - \begin{pmatrix} 0 & a_{12} & 0 & \cdots & \\ 0 & 0 & a_{23} & 0 & \cdots \\ 0 & 0 & 0 & a_{34} & 0 & \cdots \\ & & & \ddots & & \end{pmatrix} \varepsilon^{(n)} .$$

The asymptotic rate of convergence [7] is determined by

the λ values of the lambda-error matrix

$$(2.10) \qquad \mathcal{L}(\lambda) = \begin{pmatrix} \lambda a_{11} & a_{12} & 0 & \cdots & \\ \lambda a_{21} & \lambda a_{22} & a_{23} & 0 & \cdots \\ 0 & \lambda a_{32} & \lambda a_{33} & a_{34} & 0 \quad \cdots \end{pmatrix}$$

obtained by setting

$$(2.11) \qquad \det |\mathcal{L}(\lambda)| = 0 \quad .$$

Convergence is guaranteed (for infinite precision), if all the values of λ obtained from (2.11) are such that

$$(2.12) \qquad |\lambda| < 1 \quad .$$

The block simultaneous scheme

$$(2.13) \qquad \begin{pmatrix} a_{11} & 0 & \cdots & \\ 0 & a_{22} & 0 & \cdots \\ 0 & 0 & a_{33} & 0 \quad \cdots \\ & & \ddots & \end{pmatrix} U^{(n+1)} = B - \begin{pmatrix} 0 & a_{12} & 0 & \cdots & \\ a_{21} & 0 & a_{23} & 0 & \cdots \\ 0 & a_{32} & 0 & a_{34} & 0 \quad \cdots \\ & & & \ddots & \end{pmatrix} U^{(n)}$$

gives rise to the lambda-error matrix (where we have used σs instead of λs so as not to confuse the eigenvalues with those of $\mathcal{L}(\lambda)$)

$$(2.14) \qquad \mathcal{R}(\sigma) = \begin{pmatrix} \sigma a_{11} & a_{12} & 0 & \cdots & \\ a_{21} & \sigma a_{22} & a_{23} & 0 & \cdots \\ 0 & a_{32} & \sigma a_{33} & a_{34} & 0 \quad \cdots \\ & & & \ddots & \end{pmatrix} \quad .$$

Friedman [10] has shown (Theorem II) that

$$\lambda = \sigma^2 \tag{2.15}$$

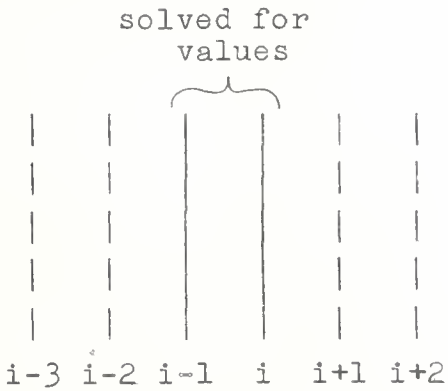
and so, if the successive scheme converges, the simultaneous scheme will also converge. We have thus been able to produce a block simultaneous iteration scheme which converges:

> Theorem 1. By a suitable partition of the matrix of elliptic difference equations it is possible to obtain convergence of the simultaneous and successive block iteration schemes.

This block, however, is rather large. In two dimensions for the bi-harmonic 13 point scheme, we would have to couple all the equations for mesh points along two neighboring lines. The coefficient matrix for this coupling would be

$$\mathcal{A}_{kk} = \begin{vmatrix} A_{kk} & A_{kk+1} \\ A_{k+1k} & A_{k+1k+1} \end{vmatrix} , \tag{2.16}$$

where the diagonal A's are five diagonal matrices and the off diagonal A's are triple diagonal matrices. The off block contributions are similarly defined. In terms of a mesh picture (for the 13 point bi-harmonic difference approximation), we are taking the contributions of the mesh points on two neighboring lines as new values (solved for values) and the remaining neighboring mesh points as old values.



In the case of the 25 point tri-harmonic difference equations in two dimensions we would need the contributions from three neighboring lines of mesh points coupled in order to obtain a simultaneous iteration scheme that would be related to the successive scheme by the relation (2.15) and hence converge. In the case of the harmonic equation in two dimensions we need only take the coupling of mesh points along one line in order to have the simultaneous and successive iteration scheme related by (2.15).

The question now arises: In two dimensions can we have convergent simultaneous block iteration schemes for elliptic difference equations which correspond to coupling along a line of mesh points only? The answer in the case of the bi-harmonic equation in two dimensions has been answered by Windsor [9], who has shown that the unextrapolated line simultaneous scheme in a rectangle will not converge when the boundary conditions that are prescribed are the function and its second normal derivative.

Whenever it is possible to relate the matrices $\mathcal{L}(\lambda)$ and $\mathcal{K}(\sigma)$ by an equivalence transformation in the form given below, the question of convergence or non-convergence of simultaneous iteration schemes is equivalent to knowing whether or not the successive scheme converges:

(2.17) $$Q(\lambda)\begin{pmatrix} \lambda A_{11} & A_{12} & A_{13} & \cdots & A_{1\,p+1} & 0 & \cdots & & \\ \lambda A_{21} & \lambda A_{22} & A_{23} & \cdots & A_{2\,p+1} & A_{2\,p+2} & 0 & \cdots & \\ \vdots & & & & & & & & \\ \lambda A_{p+1\,1} & \lambda A_{p+1\,2} & \cdots & & \lambda A_{p+1\,p+1} & A_{p+1\,p+2} & \cdots & A_{p+1\,2p+1} & 0 & \cdots \\ 0 & \lambda A_{p+2\,2} & \cdots & & & \lambda A_{p+2\,p+2} & A_{p+2\,p+3} & \cdots & A_{p+2\,2p+2} & 0 & \cdots \\ \vdots & & & & & & \ddots & & & & \\ 0 & \cdots & & & 0 & \lambda A_{n\,n-p-1} & \cdots & & & \lambda A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} f(\lambda)A_{11} & A_{12} & \cdots & & A_{1\,p+1} & 0 & \cdots & \\ A_{21} & f(\lambda)A_{22} & A_{23} & \cdots & & A_{2\,p+2} & 0 & \cdots \\ \vdots & & \ddots & & & & & \\ A_{p+1\,1} & \cdots & & A_{p+1\,p} & f(\lambda)A_{p+1\,p+1} & A_{p+1\,p+2} & \cdots & \\ 0 & A_{p+2\,2} & & & A_{p+2\,p+1} & f(\lambda) & & \\ \vdots & & & & & \ddots & & \\ 0 & & & 0 & A_{n\,n-p-1} & \cdots & A_{n\,n-1} & f(\lambda)A_{nn} \end{pmatrix},$$

where $Q(\lambda)$ and $T(\lambda)$ are lambda-matrices such that $\det |Q| \det |T| = 1$. If the transformations $Q(\lambda)$ and $T(\lambda)$ existed and $|f(\lambda)| < 1$ the simultaneous iteration scheme would converge.

Friedman [8] has proved the existence of a $Q(\lambda)$ and $T(\lambda)$ if A is of triple diagonal form. In this case $f(\lambda) = \lambda^{1/2}$. We note that in the triple diagonal case the order of correcting the equations in the simultaneous and successive schemes does not affect the asymptotic rate of convergence [9]. However, in the case of five or more diagonal systems the ordering does affect the successive scheme, yet it does not affect the simultaneous scheme. It seems impossible to reconcile this case and the generalization of Friedman's theorem to five and more diagonals may not be possible.

Hence, we will proceed to investigate the convergence of certain simultaneous schemes for all order harmonic difference equations.

3. A Class of $2p^{th}$ Order Elliptic Difference Equations.

We will consider the class of elliptic difference equations

$$(3.1) \qquad (H + V)^p \cdot U = Z$$

where

$$(3.2) \qquad H = \begin{pmatrix} H_1 & 0 & \cdots & & \\ 0 & H_2 & 0 & \cdots & \\ 0 & 0 & H_3 & 0 & \cdots \\ & & \ddots & & \\ & & & & H_\nu \end{pmatrix},$$

$$(3.3) \qquad V = \begin{pmatrix} C_1 & N_1 & 0 & \cdots & & & \\ M_2 & C_2 & N_2 & 0 & \cdots & & \\ 0 & M_3 & C_3 & N_c & 0 & \cdots & \\ \cdot & & \cdot & & & & \\ \cdot & & \cdot & & & & \\ \cdot & & \cdot & & & & \\ 0 & \cdots & & & 0 & M_\nu & C_\nu \end{pmatrix};$$

the elements of H are triple diagonal $\mu \times \mu$ matrices of scalars;

$$(3.4) \qquad H_j = \begin{pmatrix} h_{j11} & h_{j12} & 0 & \cdots & & \\ h_{j21} & h_{j22} & h_{j23} & 0 & \cdots & \\ 0 & h_{j32} & h_{j33} & h_{j34} & 0 & \cdots \\ \cdot & & \cdot & & & \\ \cdot & & \cdot & & & \\ \cdot & & \cdot & & & \\ 0 & \cdots & & & 0 & h_{j\mu\mu-1} \; h_{j\mu\mu} \end{pmatrix},$$

and the elements of V are diagonal $\mu \times \mu$ matrices:

$$(3.5) \qquad C_j = \begin{pmatrix} c_{j1} & 0 & \cdots & & \\ 0 & c_{j2} & 0 & \cdots & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & \cdots & & 0 & c_{j\mu} \end{pmatrix},$$

$$(3.6) \qquad N_j = \begin{pmatrix} n_{j1} & 0 & \cdots & & \\ 0 & n_{j2} & 0 & \cdots & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & \cdots & & 0 & n_{j\mu} \end{pmatrix},$$

$$(3.7) \qquad M_j = \begin{pmatrix} m_{j1} & 0 & \cdots & & \\ 0 & m_{j2} & 0 & \cdots & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & \cdots & & 0 & m_{j\mu} \end{pmatrix}.$$

U and Z are column vectors with $\mu \cdot \nu$ elements and p is a positive integer. This class of difference equations is suggested in analogy to the p^{th} harmonic differential equation in two dimensions where H and V are the three point difference approximations to $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ respectively. For p = 1 we would have the five point approximation to the harmonic differential operator in two dimensions; for p = 2 we would have the thirteen point approximation to the bi-harmonic differential operator; for p = 3 we would have the twenty-five point approximation to the tri-harmonic differential operator; etc.

It is for the class of difference equations (3.1) that we would like to know whether block simultaneous schemes converge. Some we know do converge: those schemes which come under the conditions of theorem I; some we know do not converge: the bi-harmonic line scheme treated by Windsor [7].

To say anything about simultaneous iteration schemes for solving (3.1) without some further restrictions on H and V, seems very difficult. We will therefore assume that H and V commute:

$$HV = VH \quad . \tag{3.8}$$

From (3.2-3.4) we find

$$H_j C_j = C_j H_j \tag{3.9}$$

(3.10) $$H_j N_j = N_j H_{j+1}$$

(3.11) $$H_{j+1} M_{j+1} = M_{j+1} H_j$$

Since C_j is diagonal, we conclude from (3.9) that

(3.12) $$C_j = \gamma_j I_\mu \quad ,$$

where γ_j is a constant and I_μ is a $\mu \times \mu$ unit matrix. (From now on we will use the subscripts μ and ν to designate the number of rows and columns in a square matrix.) The relation (3.10) shows that all H_j are similar because of the transitivity property of the similarity relationship. From (3.10-3.11) we have

(3.13) $$H_j N_j M_{j+1} = N_j M_{j+1} H_{j+1} \quad .$$

Since N_j and M_{j+1} are diagonal matrices, their product will be diagonal and commute with H_j if

(3.14) $$N_j M_{j+1} = \eta_j I_\mu$$

where η_j is a scalar. We have obtained the conditions on the elements of H and V by equating the elements of the commutator of HV. No other restrictions are necessary for H and V to commute; therefore we have

Lemma 1. H and V will commute if, and only if

(i) the diagonal blocks of V are scalar matrices:

$$C_j = \nu_j I_\mu \quad ,$$

(ii) the diagonal elements of H are similar with the similarity transformation

$$H_j = N_j H_{j+1} N_j^{-1}$$

and

(iii) the non-null off diagonal elements of V are such that

$$M_{j+1} = \eta_j N_j^{-1} \quad .$$

We are interested in difference equations derived from some sensible approximation to elliptic differential equations, and in these cases other restrictions on H and V follow. If H were derived from $K(x,y)\frac{\partial^2 w(x,y)}{\partial x^2} + L(x,y)\frac{\partial w(x,y)}{\partial x} + P(x,y)\,w(x,y)$ and V from $R(x,y)\frac{\partial^2 w(x,y)}{\partial y^2} + S(x,y)\frac{\partial w(x,y)}{\partial y} + Q(x,y)\,w(x,y)$ by three point approximations, we would find

$$\begin{aligned} Kw_{xx} + Lw_x + Mw \approx\ & (K_{ij}\ell_i + L_{ij}\lambda_i)u_{i-1\,j} \\ & - 2(K_{ij}f_i + L_{ij}b_i - \tfrac{1}{2}M_{ij})u_{ij} \\ & + (K_{ij}\gamma_i + L_{ij}\rho_i)u_{i+1\,j} \end{aligned} \tag{3.15}$$

by Taylor series approximation to the derivatives of w; i.e.

$$w_{xx} \approx \ell_i u_{i-1\,j} - 2f_i u_{ij} + \gamma_i u_{i+1\,j}$$

and similarly for w_x. The coefficients ℓ_i, f_i and γ_i and

γ_i, b_i and β_i are dependent on the mesh spacing only. From (3.15) we see that the elements of H_j are

$$h_{j\,i\,i-1} = K_{ij}\ell_i + L_{ij}\lambda_i \tag{3.16}$$

$$h_{j\,i\,i} = -2(K_{ij}f_i + L_{ij}b_i - \tfrac{1}{2}M_{ij}) \tag{3.17}$$

$$h_{j\,i\,i+1} = K_{ij}\gamma_i + L_{ij}\beta_i \quad . \tag{3.18}$$

If we multiply (3.10) out in detail, we see that the diagonal elements of H_j are independent of j:

$$h_{j\,i\,i} = h_{i\,i} \tag{3.19}$$

say; and the off diagonal elements satisfy

$$h_{u\,i-1\,i}\, h_{j\,i\,i-1} = h_{j+1\,i-1\,i}\, h_{j+1\,i-1\,i} \quad , \tag{3.20}$$

which implies $h_{j\,i-1\,i}\, h_{j\,i\,i-1}$ is independent of j. Since i and j are independent indices, we conclude from (3.17) and (3.19) that K_{ij}, L_{ij} and M_{ij} are independent of j. Then from (3.16) and (3.18) we have

$$h_{j\,i\,i-1} = h_{i\,i-1} \tag{3.21}$$

$$h_{j\,i\,i+1} = h_{i\,i+1} \quad . \tag{3.22}$$

quantities independent of j.

Thus we arrive at the result that H_j is independent of j:

$$(3.23) \qquad H_j = \begin{pmatrix} h_{11} & h_{12} & 0 & \cdots & & & \\ h_{21} & h_{22} & h_{23} & 0 & \cdots & & \\ 0 & h_{32} & h_{33} & h_{34} & 0 & \cdots & \\ & & \ddots & & & & \\ 0 & \cdots & 0 & h_{i\,i-1} & h_{ii} & h_{i\,i+1} & 0 \cdots \\ \vdots & & & & \ddots & & \end{pmatrix} = \mathcal{H}_\mu$$

We can write

$$(3.24) \qquad H = \begin{pmatrix} \mathcal{H}_\mu & 0 & \cdots \\ 0 & \mathcal{H}_\mu & 0 \cdots \\ & \ddots & \\ & & \ddots \end{pmatrix} = I_\nu{}^x \cdot \mathcal{H}_\mu$$

where we have used the direct product of two matrices notation [10].

From (3.10) and 3.23) we conclude that N_j is a scalar matrix:

$$(3.25) \qquad N_j = n_j I_\mu$$

where n_j is a constant. Finally we conclude from (3.14) that

$$(3.26) \qquad M_j = m_j I_\mu$$

where m_j is a constant. Thus we can write

$$(3.27) \qquad V = \begin{pmatrix} \nu_1 I_u & n_1 I_u & 0 & \cdots \\ m_2 I_u & \nu_2 I_u & n_2 I_u & 0 \quad \cdots \\ 0 & m_3 I_u & \nu_3 I_u & n_3 I_u \quad 0 \quad \cdots \\ & & \ddots & \end{pmatrix} = \mathcal{V}_\nu x \cdot I_u \quad ,$$

$\mathcal{V}_\nu$ being the triple diagonal matrix of scalars m_i, ν_i and n_i. Since we have found the conditions for the commuting of H and V by solving the resulting element equations of HV = VH when H and V are arrived at by three point Taylor series approximations, we have

> Theorem 2. If H and V is derived from three point Taylor series approximations to a second order linear elliptic differential operator, H and V will commute if, and only if,
>
> $$H = I_\nu x \cdot \mathcal{H}_\mu$$
>
> $$V = \mathcal{V}_\nu x \cdot I_\nu \quad .$$

In terms of a mesh picture, the region must be a rectangle and the variable coefficients in the differential operator must be functions of x only when multiplying the x derivatives and functions of y only when multiplying the y derivatives. The result of Birkhoff and Varga [11] is a special case of theorem 2: the case when the differential operators have constant coefficients.

Furthermore, we see that the boundary conditions at all the points on the left side of the rectangle are the

same because $H = I_\nu x \cdot \mathcal{H}_\mu$, i.e. the boundary conditions implied in H and therefore $\mathcal{H}_\mu$ are independent of the j index -- the numbering of the points along the left side. Similar considerations apply to the other sides. The only boundary conditions that are possible, because of the form of $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$ are even normal derivatives given or odd normal derivatives given.

4. The Line Simultaneous Iteration Scheme.

Line simultaneous iteration schemes are characterized by the mesh picture fact that a line of mesh points is corrected using old values on the surrounding lines. In terms of the class of difference equations considered in §3 we will formulate line simultaneous schemes.

Under the conditions of theorem 2 (the only cases considered for the remainder of the paper), we can write the difference equations (3.1) as

$$(4.1) \qquad \sum_{k=0}^{p} \binom{p}{k} (I_\nu x \cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x \cdot I_\mu)\, U = Z$$

since H and V commute and are of the form given by (3.24) and (3.27). For the construction of line simultaneous iteration schemes, we consider the following splitting of $\mathcal{V}_\nu^k x \cdot I_\mu$: let

$$(4.2) \qquad \mathcal{S}(\mathcal{V}_\nu^k x \cdot I_\mu) = (-)^k \max_i |v_{ii}^{(k)}|\, I_\nu x \cdot I_\mu = \mathcal{S}\mathcal{V}_\nu^k x \cdot I_\mu$$

a diagonal scalar matrix, where $v_{ii}^{(k)}$ are the diagonal elements of $\mathcal{V}_\nu^k x \cdot I_\mu$. We define the remaining part of the splitting as

$$(4.3) \qquad \mathcal{R}(\mathcal{V}_\nu^k x \cdot I_\mu) = \mathcal{V}_\nu^k x \cdot I_\mu - \mathcal{S}\mathcal{V}_\nu^k x \cdot I_\mu = \mathcal{R}\mathcal{V}_\nu^k x \cdot I_\mu \quad .$$

With these definitions of diagonal scalar matrix and remaining part matrix, we easily verify

Lemma 2. The matrices

$\mathcal{S}\mathcal{V}_\nu^k x \cdot I_\mu$, $\mathcal{S}\mathcal{V}_\nu^\ell x \cdot I_\mu$, $\mathcal{R}\mathcal{V}_\nu^k x \cdot I_\mu$ and $\mathcal{R}\mathcal{V}_\nu^\ell x \cdot I_\mu$ commute.

We now define an iteration scheme which we will call "separable line simultaneous." Using the splitting of (4.2) and (4.3), the $n+1^{st}$ iterate is given by

$$(4.4)\qquad \sum_{k=0}^{p} \binom{p}{k} (I_\nu x \cdot \mathcal{H}_\mu^{p-k}) \left[(\mathcal{S}\mathcal{V}_\nu^k x \cdot I_\mu) U^{(n+1)} + (\mathcal{R}\mathcal{V}_\nu^k x \cdot I_\mu) U^{(n)} \right] = Z \ ,$$

or in terms of the error defined by (2.8)

$$(4.5)\qquad \sum_{k=0}^{p} \binom{p}{k} (\mathcal{S}\mathcal{V}_\nu^k x \cdot \mathcal{H}_\mu^{p-k}) \varepsilon^{(n+1)} = - \sum_{k=0}^{p} \binom{p}{k} (\mathcal{R}\mathcal{V}_\nu^k x \cdot \mathcal{H}_\mu^{p-k}) \varepsilon^{(n)}$$

where we have combined the matrix products into direct products of matrices.

The matrix equation (4.5) is of the form

$$(4.6)\qquad R\varepsilon^{(n+1)} = - S\varepsilon^{(n)} \ ,$$

and if we expand the initial guess $\varepsilon^{(0)}$ into eigenvectors of $-R^{-1}S$:[2]

$$(4.7)\qquad \varepsilon^{(0)} = \sum_{\substack{r=1 \\ s=1}}^{\substack{\nu \\ \mu}} A_{rs} \, \xi^{(rs)}$$

where $\xi^{(rs)}$ is an eigenvector of $-R^{-1}S$, we see from (4.6) that

---

[2] The expansion (4.7) is possible if the elementary divisors of $R^{-1}S$ are simple. We will assume this fact and in all our examples it will be true because $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$ will be symmetric.

$$\varepsilon^{(n+1)} = \sum_{r,s} \lambda_{rs}^{n} \, A_{rs} \, \xi^{(rs)} \tag{4.8}$$

where λ_{rs} is an eigenvalue of the matrix $-R^{-1}S$ corresponding to the eigenvector $\xi^{(rs)}$.

Substituting (4.8) into (4.5), we find that each eigenvector satisfies the equation

$$\sum_{k=0}^{p} \binom{p}{k} \left[\lambda_{rs} \; \mathcal{S}\mathcal{V}_{\nu}^{k} \cdot \mathcal{H}_{\mu}^{p-k} + \mathcal{R}\mathcal{V}_{\nu}^{k} \cdot \mathcal{H}_{\mu}^{p-k} \right] \xi^{(rs)} = 0 \quad . \tag{4.9}$$

We can further simplify (4.9) because of lemma 2, and the fact that $\mathcal{H}_{\mu}$ and its powers commute. Since all the $\mu \times \mu$ matrices commute, they have common eigenvectors; since all the $\nu \times \nu$ matrices commute, they have common eigenvectors; since the matrices in (4.9) are in directed product form, we can write

$$\xi^{(rs)} = \xi_{\nu}^{(s)} \cdot \xi_{\mu}^{(r)} \tag{4.10}$$

where the subscripts μ and ν indicate the number of rows in the vectors $\xi_{\mu}^{(r)}$ and $\xi_{\nu}^{(s)}$ respectively, and where $\xi_{\nu}^{(s)}$ is a common eigenvector of the $\nu \times \nu$ matrices of (4.9) and $\xi_{\mu}^{(r)}$ is a common eigenvector of the $\mu \times \mu$ matrices of (4.9). Thus

$$\mathcal{S}\mathcal{V}_{\nu}^{k} \, \xi_{\nu}^{(s)} = \sigma_{s}^{(k)} \, \xi_{\nu}^{(s)} \quad , \tag{4.11}$$

$$\mathcal{R}\mathcal{V}_{\nu}^{k} \, \xi_{\nu}^{(s)} = \rho_{s}^{(k)} \, \xi_{\nu}^{(s)} \quad , \tag{4.12}$$

and

$$\mathcal{H}_\mu \, \xi_\mu^{(r)} = h_r \, \xi_\mu^{(r)} \quad , \tag{4.13}$$

where $\sigma_s^{(k)}$, $\rho_s^{(k)}$ and h_r are the eigenvalues of the matrices $\mathcal{S}\mathcal{V}_\nu^k$, $\mathcal{R}\mathcal{V}_\nu^k$ and $\mathcal{H}_\mu$. Substituting (4.10) into (4.9) and using (4.11-4.13), we find

> Theorem 3. The eigenvalues of the error matrix for the "separable line simultaneous" iteration scheme given by (4.4) are

$$\lambda_{rs} = - \frac{\sum_{k=0}^{p} \binom{p}{k} \rho_s^{(k)} \, h_r^{p-k}}{\sum_{k=0}^{p} \binom{p}{k} \sigma_s^{(k)} \, h_r^{p-k}} \quad . \tag{4.14}$$

It is interesting to note that we have said very little about the matrices $\mathcal{V}_\nu$ and $\mathcal{H}_\mu$, except that they are triple diagonal. Knowing the eigenvalues of the $\nu \times \nu$ and the $\mu \times \mu$ matrices in (4.4) is sufficient to find the eigenvectors of the $\mu \cdot \nu \times \mu \cdot \nu$ matrix $R^{-1}S$. In terms of a mesh picture we have found a large class of two-dimensional iteration schemes whose error matrix's eigenvalues λ_{rs} are evaluable in terms of rational functions of eigenvalues $\sigma_s^{(k)}$, $\rho_s^{(k)}$ and h_r of one-dimensional operators $\mathcal{S}\mathcal{V}_\nu^k$, $\mathcal{R}\mathcal{V}_\nu^k$ and $\mathcal{H}_\mu$.

We have called the scheme given by (4.4) "separable line simultaneous." The designation "separable" is clear in view of (4.10); the designation "line simultaneous" is not

clear. We will clarify "line simultaneous" by a series of examples in the next section.

5. Examples of "Separable Line Simultaneous" Iteration Schemes.

The case of p = 1 is the so-called "line Richardson" scheme [4]. For equal spacing $\triangle x$ in the i direction (x or horizontal direction) and equal spacing $\triangle y$ in the j direction (y or vertical direction) and for the case when the unknown is prescribed on the rectangular boundary, we have

$$(5.1) \qquad \mathcal{H}_\mu = \frac{1}{(\triangle x)^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & & & \\ -1 & 2 & -1 & 0 & \cdots & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & & & 0 & -1 & 2 \end{pmatrix}$$

$$(5.2) \qquad \mathcal{V}_\nu = \frac{1}{(\triangle y)^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & & & \\ -1 & 2 & -1 & 0 & \cdots & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & \\ \vdots & & & \ddots & & & \\ 0 & \cdots & & & 0 & -1 & 2 \end{pmatrix}$$

and the equation for a component of the error is (cf. (4.9))

$$(5.3) \qquad \lambda_{rs}(\mathcal{A}\mathcal{V}_\nu^0 x \cdot \mathcal{H}_\mu + \mathcal{A}\mathcal{V}_\nu x \cdot \mathcal{H}_\mu)\xi^{(rs)}$$

$$= -(\mathcal{R}\mathcal{V}_\nu^0 x \cdot \mathcal{H}_\mu + \mathcal{R}\mathcal{V}_\nu x \cdot \mathcal{H}_\mu)\xi^{(rs)} \quad .$$

We easily verify that the eigenvector $\xi^{(rs)}$ is (see (4.10))

$$(5.4)\qquad \xi_\nu^{(s)} \times \xi_\mu^{(r)} = \begin{pmatrix} \sin\frac{s\pi}{\nu+1} \\ \sin\frac{2s\pi}{\nu+1} \\ \vdots \\ \sin\frac{js\pi}{\nu+1} \\ \vdots \\ \sin\frac{\nu s\pi}{\nu+1} \end{pmatrix} \times \begin{pmatrix} \sin\frac{r\pi}{\mu+1} \\ \sin\frac{2r\pi}{\mu+1} \\ \vdots \\ \sin\frac{ir\pi}{\mu+1} \\ \vdots \\ \sin\frac{\mu r\pi}{\mu+1} \end{pmatrix} ;$$

for with $\mathcal{H}_\mu$ given by (5.1) and $\xi_\mu^{(r)}$ given by (5.4), we have

$$(5.5)\qquad \mathcal{H}_\mu \xi_\mu^{(r)} = \frac{2}{(\triangle x)^2}\left(1 - \cos\frac{r\pi}{\mu+1}\right)\xi_\mu^{(r)} = h_r \xi_\mu^{(r)} ;$$

for with $\mathcal{V}_\nu$ given by (5.2) and $\xi_\nu^{(s)}$ given by (5.4), we have

$$(5.6)\qquad \mathcal{V}_\nu^0\, \xi_\nu^{(s)} = \frac{-1}{(\triangle y)^2}\, \xi_\nu^{(s)} = \sigma_s^{(0)}\, \xi_\nu^{(s)} ,$$

$$(5.7)\qquad \mathcal{V}_\nu\, \xi_\nu^{(s)} = \frac{2}{(\triangle y)^2}\, \xi_\nu^{(s)} = \sigma_s^{(1)}\, \xi_\nu^{(s)} ,$$

$$(5.8)\qquad \mathcal{V}_\nu^0\, \xi_\nu^{(s)} = 0 \cdot \xi_\nu^{(s)} = \rho_s^{(0)}\, \xi_\nu^{(s)} ,$$

and

$$(5.9)\qquad \mathcal{V}_\nu\, \xi_\nu^{(s)} = \frac{-2}{(\triangle y)^2} \cos\frac{s\pi}{\nu+1}\, \xi_\nu^{(s)} = \rho_s^{(1)}\, \xi_\nu^{(s)} .$$

From theorem 3 the eigenvalues of the error matrix for the "separable line simultaneous" scheme (5.3) with p = 1 are

$$(5.10) \qquad \lambda_{rs} = \frac{\frac{1}{(\triangle y)^2}\cos\frac{s\pi}{\nu+1}}{\frac{1}{(\triangle x)^2}\cos\frac{r\pi}{\mu+1} - \left(\frac{1}{(\triangle x)^2} + \frac{1}{(\triangle y)^2}\right)}$$

the result found in the literature (cf. [4]).

For a general p and $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$ given by (5.1-.2) we would be approximating the p^{th} harmonic differential equation with the function and the first p even normal derivatives given on the rectangular boundary. Again a component of the error is given by (5.4). We have the $\nu \times \nu$ matrix

$$(5.11) \qquad \mathcal{V}_\nu^k = \frac{-1}{(\triangle y)^{2k}} \left((-)^{k+c-r}\binom{2k}{k+c-r} - (-)^{k-c-r}\binom{2k}{k-c-r} \right),$$

$$r,c = 1,2,\ldots,\nu$$

where it is understood that

$$(5.12) \qquad \binom{2k}{\tau} = \frac{(2k)!}{!(2k-\tau)!} \qquad \text{if} \quad 0 \leq \tau \leq 2k$$

$$= 0 \qquad \text{if} \quad 2k < \tau \quad \text{or} \quad \tau < 0 \quad .$$

For k = 1, (5.11) agrees with (5.2) and by induction we establish the $\mathcal{V}_\nu^k$ is as given by (5.11).

The scalar matrix

$$(5.13) \qquad \mathcal{S}\mathcal{V}_\nu^k = \frac{-1}{(\triangle y)^{2k}} (-)^k \binom{2k}{k} I_\nu$$

and the remainder matrix

(5.14)

$$\mathcal{R}\mathcal{V}_{\nu}^{k} = \frac{-1}{(\triangle y)^2}\left((-)^{k+c-r}\binom{2k}{x+c-r} - (-)^{k-c-r}\binom{2k}{x-c-r} - (-)^{k}\binom{2k}{k}\delta_{rc}\right),$$

where δ_{rc} is the Kroneker delta. The eigenvectors of the scalar matrix $\mathcal{S}\mathcal{V}_{\nu}^{k}$ are given by (5.4); the eigenvalues of the scalar matrix (5.13) are

$$\sigma_{s}^{(k)} = \frac{1}{(\triangle y)^{2k}}\,(-)^{k+1}\binom{2k}{k} \quad . \tag{5.15}$$

The eigenvectors of the remainder matrix are given by (5.4); the eigenvalues of the remainder matrix (5.14) are

$$\rho_{s}^{(k)} = \frac{-1}{(\triangle y)^{2k}}\sum_{\chi=0}^{k}\binom{k}{\chi}(-2)^{k-\chi}\sum_{\substack{\tau=0\\ \neq \frac{\chi}{2}}}^{\chi}\binom{\chi}{\tau}\cos\,(\chi-2\tau)\frac{s\pi}{\nu+1} \quad . \tag{5.16}$$

Using theorem 3, we obtain the eigenvalues λ_{rs} of the "separable line simultaneous" error matrix:

$$-\lambda_{rx} = \tag{5.17}$$

$$\frac{\displaystyle\sum_{k=0}^{p}\binom{p}{k}\left\{\frac{2}{(\triangle x)^2}\left(\cos\frac{r\pi}{\mu+1}-1\right)\right\}^{p-k}\frac{1}{(\triangle y)^{2k}}\sum_{\chi=0}^{k}\binom{k}{\chi}(-2)^{k-}\sum_{\substack{\tau=0\\ \neq \frac{\chi}{2}}}^{\chi}\binom{\chi}{\tau}\cos(\chi-2\tau)\frac{s\pi}{\nu+1}}{\displaystyle\sum_{k=0}^{p}\binom{p}{k}\left\{\frac{2}{(\triangle x)^2}\left(\cos\frac{r\pi}{\mu+1}-1\right)\right\}^{p-k}\frac{1}{(\triangle y)^{2k}}(-)^{k}\binom{2k}{k}} \quad .$$

For $p = 2$ and $\triangle x = \triangle y = 1$, we obtain

$$(5.18) \qquad -\lambda_{rx} = \frac{4 \cos\frac{r\pi}{\mu+1} \cos\frac{s\pi}{\nu+1} - 8 \cos\frac{s\pi}{\nu+1} + 2 \cos^2\frac{s\pi}{\nu+1} - 1}{2 \cos^2 \frac{r\pi}{\mu+1} - 8 \cos \frac{r\pi}{\mu+1} + 9} ,$$

which is the result Windsor [6] obtains and calls the "line simultaneous displacement" scheme.

It is interesting to note that λ_{rs} takes on its upper bound when $\cos \frac{r\pi}{\mu+1} \to 1$ and $\cos \frac{s\pi}{\nu+1} \to 1$, and its lower bound when $\cos \frac{r\pi}{\mu+1} \to 1$ and $\cos \frac{s\pi}{\nu+1} \to -1$. These values are approached for an r,s when the mesh becomes very fine. From (5.17) by simply summing the combinatorial series with $\cos \frac{r\pi}{\mu+1} = \cos \frac{s\pi}{\nu+1} = 1$, we find

$$(5.19) \qquad \sup_{\substack{\mu,\nu,\\ r,s}} \lambda_{rs} = 1 .$$

From (5.17) by summing the combinatorial series with $\cos \frac{r\pi}{\mu+1} = 1$ and $\cos \frac{s\pi}{\nu+1} = -1$, we find

$$(5.20) \qquad \inf_{\substack{\mu,\nu,\\ r,s}} \lambda_{rs} = 1 - \frac{2^{2p}}{\binom{2p}{p}} .$$

For $p = 2$

$$(5.21) \qquad \inf_{\substack{\mu,\nu,\\ r,s}} \lambda_{rs} \ (p=2) = -\frac{5}{3}$$

the result found by Windsor [6].

We are thus justified in calling the scheme given by (4.4) "separable line simultaneous."

Except for p = 1, the case which comes under theorem 1, the "separable line simultaneous" schemes diverge. However, if we were to extrapolate with an extrapolation constant $\alpha < 1$, we could obtain a convergent scheme; this type of extrapolation will increase the lower bound keeping the upper bound below one [12]. For p = 2 this result was found by Windsor [6].

There are other possibilities for which we can explicitly find λ_{rs}. If there is a symmetry (even normal derivatives given) condition on the top boundary, we would have the difference equations for the approximation of the p^{th} harmonic differential equation given by (5.1) and

$$(5.22) \qquad \gamma_\nu = \frac{1}{(\triangle y)^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & & & \\ -1 & 2 & -1 & 0 & \cdots & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & \\ \vdots & & & & \ddots & & \\ 0 & \cdots & & & 0 & -2 & 2 \end{pmatrix} .$$

The eigenvectors

$$(5.23)\qquad \xi^{(rs)} = \begin{pmatrix} \sin\frac{s\pi}{2\nu} \\ \sin\frac{2s\pi}{2\nu} \\ \vdots \\ \sin\frac{js\pi}{2\nu} \\ \vdots \\ \sin\frac{\nu s\pi}{2\nu} \end{pmatrix} \times \begin{pmatrix} \sin\frac{r\pi}{\mu+1} \\ \sin\frac{2r\pi}{\mu+1} \\ \vdots \\ \sin\frac{ir\pi}{\mu+1} \\ \vdots \\ \sin\frac{\mu r\pi}{\mu+1} \end{pmatrix}$$

$$r = 1,2,\ldots,\mu$$
$$s = 1,3,5,\ldots,2\mu+1$$

and the eigenvalues λ_{rs} can be computed in terms of the eigenvalues of $\mathcal{H}_\mu$ given by (5.1) and the eigenvalues of $\mathcal{S}\mathcal{V}_\nu^k$ and $\mathcal{R}\mathcal{V}_\nu^k$ easily computable from (5.22). Other explicit examples corresponding to symmetry on other boundaries can easily be determined.

6. Separable Line Successive Iteration Schemes.

Because of our inability to extend Friedman's theorem connecting the asymptotic rates of convergence of all line simultaneous and line successive schemes, we will investigate explicit solutions of the class of equations given by (4.1) with a view of partially understanding the relationship between line simultaneous and line successive schemes.

Line successive schemes are characterized by the solving of all the equations coupled along a line (a one dimensional problem) using corrected neighboring contributions to the equations whenever available. In order to formulate this type of iteration scheme, we define the following decomposition of the vertical difference operator

$$(6.1) \qquad \mathcal{V}_\nu^k x \cdot I_\mu = \mathcal{L}(\mathcal{V}_\nu^k x \cdot I_\mu) + \mathcal{S}\mathcal{V}_\nu^k x \cdot I_\mu + u(\mathcal{V}_\nu^k x \cdot I_\mu)$$

where $\mathcal{S}\mathcal{V}_\nu^k x \cdot I_\mu$ is given as before by (4.2), the lower matrix as a matrix with zeros above the diagonal and the rest of the matrix unchanged:

$$(6.2) \qquad \mathcal{L}(\mathcal{V}_\nu^k x \cdot I_\mu) = \begin{pmatrix} v_{11}^{(k)} & 0 & \cdots & & \\ v_{21}^{(k)} & v_{22}^{(k)} & 0 & \cdots & \\ v_{31}^{(k)} & v_{32}^{(k)} & v_{33}^{(k)} & 0 & \cdots \\ \vdots & & & \ddots & \end{pmatrix} x \cdot I_\mu$$

$$= \mathcal{L}\mathcal{V}_\nu^k x \cdot I_\mu$$

and the upper matrix as that matrix with zeros on and below the diagonal and the remaining elements of the matrix unchanged:

$$(6.3)\qquad u(\mathcal{V}_\nu^k x\cdot I_\mu) = \begin{pmatrix} 0 & v_{12}^{(k)} & v_{13}^{(k)} & \cdots \\ 0 & 0 & v_{23}^{(k)} & v_{24}^{(k)} & \cdots \\ 0 & 0 & 0 & v_{34}^{(k)} & \cdots \\ \vdots & & & & \ddots \end{pmatrix} x\cdot I_\mu = u\mathcal{V}_\nu^k x\cdot I_\mu \quad .$$

We now define "separable successive line" iteration schemes as those schemes in which a component of the error $\xi^{(rs)} = \xi_\nu^{(r)} x\cdot\xi_\mu^{(s)}$ satisfies

(6.4)

$$\sum_{k=0}^{p}\binom{p}{k}\left[\underline{\lambda}_{rs}(h\mathcal{V}_\nu^k x\cdot\mathcal{H}_\mu^{p-k} + \delta\mathcal{V}_\nu^k x\cdot\mathcal{H}_\mu^{p-k}) + u\mathcal{V}_\nu^k x\cdot\mathcal{H}_\mu^{p-k}\right]\xi_\nu^{(s)} x\cdot\xi_\mu^{(r)} = 0 \quad .$$

Since $\mathcal{H}_\mu$ and its powers only operate on $\xi_\mu^{(r)}$, and since $\delta\mathcal{V}_\nu^k$ has eigenvalues given by (4.11), (6.4) reduces to

$$(6.5)\qquad \sum_{k=0}^{p}\binom{p}{k} h_r^{p-k}\left[\underline{\lambda}_{rx}(h\mathcal{V}_\nu^k + \sigma_s^{(k)} I_\nu) + u\mathcal{V}_\nu^k\right]\xi_\nu^{(s)} = 0 \quad .$$

which is a set of equations, each of the order of (in the mesh picture) a one-dimensional problem. The matrices $h\mathcal{V}_\nu^k$ and $u\mathcal{V}_\nu^k$ do not have a common set of eigenvalues; therefore we cannot in general simplify (6.5) and solve

for the eigenvalues λ_{rs} of the error matrix in terms of the eigenvalues of $\mathcal{A}\mathcal{V}_\nu^k$ and $u\mathcal{V}_\nu^k$. We can state no more than

Theorem 4. The eigenvalues λ_{rs} of the error matrix for a "separable line successive" iteration scheme satisfy the eigenvalue equation

$$(6.6) \qquad \Big(\sum_{k=0}^{p} \binom{p}{k} h_r^{p-k} u\mathcal{V}_\nu^k\Big)\xi_\nu^{(s)}$$

$$= -\lambda_{rs}\Big(\sum_{k=0}^{p} \binom{p}{k} h_r^{p-k}\left\{\mathcal{A}\mathcal{V}_\nu^k + \sigma_s^{(k)} I_\nu\right\}\Big)\xi_\nu^{(s)} .$$

7. Examples of "Separable Line Successive" Iteration Schemes.

The case of p = 1 is the so-called "line Liebmann" scheme [4] for the harmonic difference equations. In this case the eigenvalues of the error matrix and eigenvectors satisfy the equations (see (6.6))

$$h_r \xi_\nu^{(s)} = -\lambda_{rs}\left[h_{rs}(I_\nu + \sigma_s^{(0)}) + (\hbar\nu_\nu + \sigma_s^{(1)} I_\nu)\right]\xi_\nu^{(s)} \quad . \tag{7.1}$$

The eigenvectors $\xi_\nu^{(s)}$ are of the form

$$\xi_\nu^{(s)} = \begin{pmatrix} a \sin \frac{s\pi}{\nu+1} \\ a^2 \sin \frac{2s\pi}{\nu+1} \\ \vdots \\ a^j \sin \frac{js\pi}{\nu+1} \\ \vdots \\ a^\nu \sin \frac{\nu s\pi}{\nu+1} \end{pmatrix} \tag{7.2}$$

where

$$a = \lambda_{rs}^{1/2} \quad . \tag{7.3}$$

The relation (7.3) is easily determined from (7.1); a solution of a quadratic equation is involved in obtaining (7.3). From (7.1-.3), (6.2-.3) and (5.1-.2) we obtain

$$\lambda_{rs} = \left(\frac{\frac{1}{(\triangle y)^2} \cos \frac{s\pi}{\nu+1}}{\frac{1}{(\triangle x)^2} \cos \frac{r\pi}{\mu+1} - \left(\frac{1}{(\triangle x)^2} + \frac{1}{(\triangle y)^2}\right)} \right)^2 \tag{7.4}$$

which is the square of the eigenvalues of the corresponding "separable line simultaneous" scheme. This, of course, is the value for λ_{rs} that is found in the literature [4].

For $p > 1$ (7.2) is still the form of the eigenfunction for the equation (6.6), but a is not given by (7.3) and in trying to find a, we are led to a polynomial equation for a of order 2p. Except for $p = 2$, this polynomial equation cannot be solved explicitly and in the case of $p = 2$ the result is very messy and will not be exhibited here. This, possibly, accounts for the fact that no solutions of these equations, other than the case $p = 1$, are found in the literature.

8. Alternating Direction Schemes.

There are currently two types of alternating direction schemes described in the literature: the Peaceman Rackford [13] (PR) and Douglas Rackford [14] (DR) schemes.

The PR scheme is a special case of

(8.1)

$$\sum_{k=0}^{q}\binom{p}{k}(I_\nu x \cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x \cdot I_\mu)U^{(n+\frac{1}{2})} + \sum_{k=q+1}^{p}\binom{p}{k}(I_\nu x \cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x \cdot I_\mu)U^{(n)}$$

$$= Z + \omega_n(U^{(n+\frac{1}{2})} - U^{(n)})$$

(8.2)

$$\sum_{k=0}^{q}\binom{p}{k}(I_\nu x \cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x \cdot I_\mu)U^{(n+\frac{1}{2})} + \sum_{k=q+1}^{p}\binom{p}{k}(I_\nu x \cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x \cdot I_\mu)U^{(n+1)}$$

$$= Z + \omega_n(U^{(n+1)} - U^{(n+\frac{1}{2})}) \quad ,$$

where ω_n are suitably chosen parameters which hasten the reduction of errors, and q is an integer between 0 and p. When q = 0 and p = 1, we obtain the scheme originally treated by Peaceman and Rackford [13].

As p increases, there are a multitude of possible schemes. However, in each of these schemes it is not possible to "simply" solve the resulting coupled difference

equations. The quantity 2q + 1 is the number of lines of points coupled on each alternating sweep: for the original PR scheme, q = 0, the number of lines coupled is 1. The success of the PR scheme for p > 1 depends on two conditions: (i) the choice of a set of ω_n guaranteeing good reduction of errors and (ii) the possibility of coupling one line of points so that a matrix of small order need be inverted.

The DR scheme uses the difference scheme derived from the differential equations on the first sweep, but on the second sweep a relationship between parts of the difference operators are used, which guarantees reduction of errors. A generalization of the DR scheme is given by (8.1) and

$$(8.3)\quad \sum_{k=q+1}^{p} \binom{p}{k} (I_\nu x \cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x \cdot I_\mu) U^{(n+1)}$$

$$= \sum_{k=q+1}^{p} \binom{p}{k} (I_\nu x \cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x \cdot I_\mu) U^{(n)} + \omega_n (U^{(n+\frac{1}{2})} - U^{(n)})$$

When p = 1 and q = 0, we obtain the scheme first treated by Douglas and Rackford [14]; when p = 2 and q = 0, we obtain the alternating direction treatment of the biharmonic given by Conte and Dames [15].

Defining the error as in (2.8), we obtain the error equation for the PR scheme from (8.1) and (8.2) as

$$(8.4)\qquad \varepsilon^{(n+1)} = \Big[\sum_{k=q+1}^{p} \binom{p}{k}(I_\nu x\cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x\cdot I_\mu) - W_n\Big]^{-1}\cdot$$

$$\Big[\sum_{k=0}^{q} \binom{p}{k}(I_\nu x\cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x\cdot I_\mu) + W_n\Big]\cdot$$

$$\Big[\sum_{k=0}^{q} \binom{p}{k}(I_\nu x\cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x\cdot I_\mu) - W_n\Big]^{-1}\cdot$$

$$\Big[\sum_{k=q+1}^{p} \binom{p}{k}(I_\nu x\cdot \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x\cdot I_\mu) + W_n\Big]\cdot\varepsilon^{(n)} \quad .$$

Since $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$ commute, we can expand the error as in (4.8) and (4.10):

$$(8.5)\qquad \varepsilon^{(n)} = \sum_{r,s} A_{rs}^{(0)}\ \lambda_{rs}^n\ \xi_\nu^{(s)} x\cdot \xi_\mu^{(r)}$$

where λ_{rs} are the eigenvalues of the matrix in (8.4). If

$$(8.6)\qquad \mathcal{V}_\nu\ \xi_\nu^{(s)} = v_s\ \xi_\nu^{(s)} \quad ,$$

we obtain from (8.4), (8.5) and (4.13) the eigenvalues λ_{rs}:

Theorem 5. The eigenvalues of the error matrix for the PR alternating direction scheme are

$$(8.7)\qquad \lambda_{rs}^{(PR)} = \frac{\Big(\sum_{k=0}^{q} \binom{p}{k} h_r^{p-k}v_s^k + W_n\Big)\Big(\sum_{k=q+1}^{p} \binom{p}{k} h_r^{p-k}v_s^k + W_n\Big)}{\Big(\sum_{k=q+1}^{p} \binom{p}{k} h_r^{p-k}v_s^k - W_n\Big)\Big(\sum_{k=0}^{q} \binom{p}{k} h_r^{p-k}v_s^k - W_n\Big)}$$

In a similar manner we find the eigenvalues λ_{rs} for the DR scheme:

Theorem 6. The eigenvalues of the error matrix for the DR alternating direction scheme are

$$(8.8)\quad \lambda_{rs}^{(DR)} = \frac{\sum_{k=0}^{q} \binom{p}{k} h_r^{p-k} v_s^k \sum_{k=q+1}^{p} \binom{p}{k} h_r^{p-k} v_s^k + w_n^2}{\sum_{k=0}^{q} \binom{p}{k} h_r^{p-k} v_s^k \sum_{k=q+1}^{p} \binom{p}{k} h_r^{p-k} v_s^k + w_n^2 - w_n (h_r + v_s)^p} .$$

There are many other alternating direction type schemes; for example, a set of q_m can be given in (8.1) and (8.2) or (8.1) and (8.3). It is an easy exercise to find the error matrices and their corresponding eigenvalues in terms of the eigenvalues of $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$. We will consider only one other scheme which we will call the "simple alternating direction scheme (S). The first sweep is the same as the PR scheme (8.1) and the second alternating sweep is

$$(8.9)\quad \sum_{k=q+1}^{p} \binom{p}{k} (I_\nu x \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x I_\mu) U^{(n+\frac{1}{2})}$$

$$+ \sum_{k=0}^{q} \binom{p}{k} (I_\nu x \mathcal{H}_\mu^{p-k})(\mathcal{V}_\nu^k x I_\mu) U^{(n+1)} = Z + w_n (U^{(n+1)} - U^{(n+\frac{1}{2})}) .$$

As before the eigenvalues are easily determined in terms of the eigenvalues of $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$. We have

Theorem 7. The eigenvalues of the error matrix for the S alternating direction scheme are

$$\lambda_{rs}^{(S)} = \frac{\left(\sum_{k=q+1}^{p} \binom{p}{k} h_r^{p-k} v_s^k + w_n \right)^2}{\left(\sum_{k=0}^{q} \binom{p}{k} h_r^{p-k} v_s^k - w_n \right)^2} \quad . \tag{8.10}$$

9. Eigenvalues and Eigenvectors of Some Alternating Direction Schemes.

The first alternating direction scheme was that exhibited by Peaceman and Rackford [13]. In our terminology $q = 0$ and $p = 1$. If we consider the case of Dirichlet data on the boundary, $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$ are given by (5.1) and (5.2). The eigenvectors of the error matrix are

$$(9.1) \qquad \xi^{(rs)} = \begin{pmatrix} \sin \frac{s\pi}{\nu+1} \\ \sin \frac{2s\pi}{\nu+1} \\ \vdots \\ \sin \frac{js\pi}{\nu+1} \\ \vdots \\ \sin \frac{\nu s\pi}{\nu+1} \end{pmatrix} \times \begin{pmatrix} \sin \frac{r\pi}{\mu+1} \\ \sin \frac{2r\pi}{\mu+1} \\ \vdots \\ \sin \frac{ir\pi}{\mu+1} \\ \vdots \\ \sin \frac{\mu r\pi}{\mu+1} \end{pmatrix}, \qquad \begin{matrix} r = 1,2,\ldots,\mu \\ s = 1,2,\ldots,\nu \end{matrix}$$

and the eigenvalues given by theorem 5 are

$$(9.2) \qquad \lambda_{rs}^{(PR)}(p=1;q=0) = \frac{(h_r + w_n)(v_s + w_n)}{(v_s - w_n)(h_r - w_n)}$$

The eigenvalues h_r are given by (5.5) while the eigenvalues v_s are given by a similar expression. The choice of $w_n < 0$ will ensure $|\lambda_{rs}^{(PR)}| < 1$ because h_r and $v_s > 0$, since they are eigenvalues of the positive definite matrices $\mathcal{H}_\mu$ and $\mathcal{V}_\nu$. The success of this PR scheme depends on a choice of $w_n = -h_r$ for each r.

The Douglas Rackford [14] scheme in our terminology is obtained when q = 0 and p = 1. The eigenvectors are given by (9.1), and the eigenvalues of the error matrix are given by theorem 6:

$$(9.3) \qquad \lambda_{rs}^{(DR)}(p=1;q=0) = \frac{v_s h_r + W_n^2}{v_s h_r + W_n^2 - W_n(h_r + v_s)} \quad .$$

Since v_s and $h_r > 0$, a choice of $W_n < 0$ will ensure $|\lambda_{rs}^{(DR)}(p=1;q=0)| < 1$ for all r and s.

Recently a scheme for the solution of the biharmonic has been proposed by Conte and Dames [15]. This scheme is of the DR type with q = 0 and p = 2.[3] The eigenvectors of the error matrix are given by (9.1) and the eigenvalues are given by theorem 6:

$$(9.4) \qquad \lambda_{rs}^{(DR)}(p=2;q=0) = \frac{(2h_r v_s + v_s^2)h_r^2 + W_n^2}{(2h_r v_s + v_s^2)h_r^2 + W_n^2 - W_n(h_r + v_s)^2}$$

In this case a choise of $W_n < 0$ will ensure $|\lambda_{rs}^{(DR)}(p=2;q=0)| < 1$ for all r and s. In general a choice of $(-)^p W_n < 0$ will ensure convergence of the DR scheme as can easily be seen from theorem 6.

---

[3] In a private discussion Dr. S. Conte expressed the opinion that their treatment of the biharmonic was a generalization of the DR scheme and not the PR scheme.

A possible PR treatment of the biharmonic equation would be q = 0 and p = 2. In this case we find from theorem 5:

$$(9.5) \qquad \lambda_{rs}^{(PR)}(p=2;q=0) = \frac{(h_r^2 + W_n)(2h_r v_s + v_s^2 + W_n)}{(h_r^2 - W_n)(2h_r v_s + v_s^2 - W_n)} \quad .$$

We can choose $W_n < 0$ and obtain convergence. However, the work required to solve the difference equations for the new values on the second sweep -- equations (8.2) -- is almost as much work as solving the original difference equations. On the second sweep each line of points is coupled with the neighboring lines of points, which in turn is coupled with their neighboring lines, ... until we arrive at the boundary.

The S alternating direction scheme with p = 2 and q = 0 affords us a way out of this difficulty. In this case the eigenvalues of the error matrix are

$$(9.6) \qquad \lambda_{rs}^{(S)}(p=2;q=0) = \frac{(2h_r v_s + v_s^2 + W_n)^2}{(h_r^2 - W_n)^2} \quad .$$

For convergence of this iteration scheme we must have $|\lambda_{rs}^{(S)}(p=2;q=0)| < 1$. Hence

$$(9.7) \qquad |2h_r v_s + v_s^2 + W_n| < |h_r^2 - W_n|$$

which leads to the condition

$$(9.8) \qquad W_n < h_r^2 - \tfrac{1}{2}(h_r + v_s)^2 \quad .$$

This inequality must be satisfied for all r and s. Thus the set of W_n must be chosen such that all

$$(9.9) \qquad W_n < \min_{r,s} \left\{ h_r^2 - \tfrac{1}{2}(h_r + v_s)^2 \right\} \quad .$$

It is not possible to choose W_n in a manner which will allow some of the eigenvalues of the error matrix to vanish on each iteration. Only in the PR alternating direction scheme is it possible to choose a set of W_n such that some of the eigenvalues of the error matrix vanish for each iteration. These results easily follow from theorems 5, 6 and 7.

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